Spin transport in semiconductors: the spin drag and spin Hall effects

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Outline

I. Introduction to spin transport:
key differences between spin currents and charge currents

spin Hall and spin drag effects
Z2 topology of band insulators

II. Looking ahead: quantum information with spin

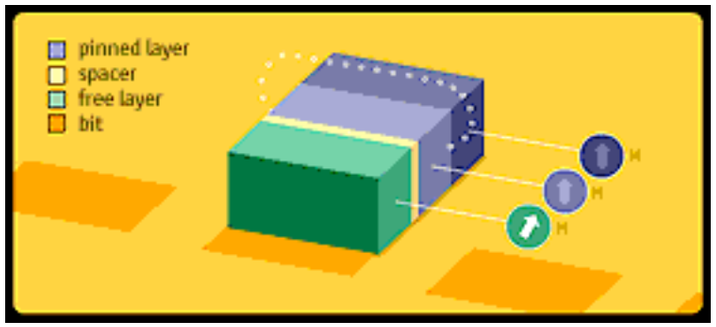
How much "entanglement" is generated by local spin Hamiltonians?

The work discussed was completed with

Cenke Xu (UCB), Gil Refael (Caltech)
J. W. Orenstein, C. P. Weber, N. Gedik (UCB)
J. Stephens, D. D. Awschalom, L. Balents (UCSB)

Spintronics: present

- Magnetic storage is already an established industry
- A magnetic domain on a hard disk has many spins
- However, the quantum nature of single spins is already used in some devices, such as "giant magnetoresistance" spin-valve read heads



Spin changes across junction: high R
Spin constant across junction: low R
(figure from IBM)

- The goal of "spintronics" is to replace many types of charge-based electronics with spin. Possible advantages are lower power consumption and smaller device size (Si charge limit: c. 2025).

Spintronics: future

- Medium-term goal: transistors in which *spin currents* rather than *charge currents* perform classical logic operations
- Long-term goal: coherent spin manipulation for quantum logic operations ("quantum computing")

Note: Optical measurements of spin coherence use techniques similar to those discussed in this talk

Quantum computing requires being able to *create* and *manipulate* superpositions: difficult, but extremely powerful

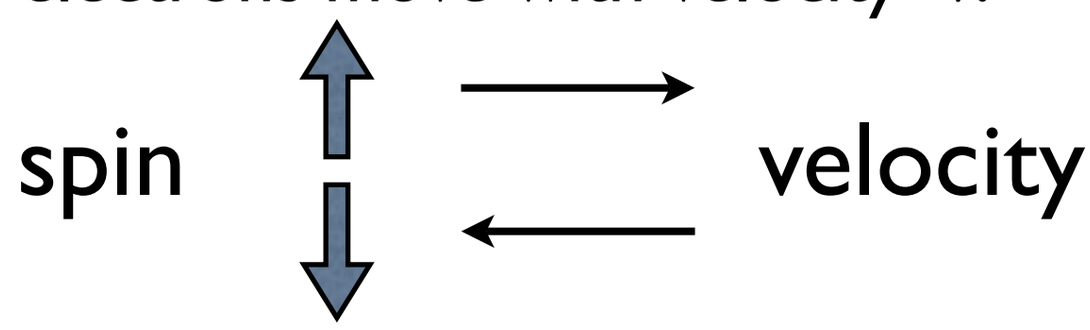
Classical computing with spin requires only "up" and "down" states—much less demanding

What is a spin *current*?

Intuitive examples:

a *spin-polarized* current of electrons is one in which all electrons are in the same spin state: this current carries both charge and spin.

Example of a pure spin current:
suppose *n* spin-up electrons move with velocity *v*,
and *n* spin-down electrons move with velocity -*v*.



Then there is zero net charge current, but there is a current of spin:

More formally, spin current carries two vector indices

$\mathcal{J}^i_j$ = current of spin direction *i* in spatial direction *j*

Science of spin currents

There are some key differences between spin and charge currents

Perhaps the most fundamental is that spin, unlike charge, is not conserved:

Spin is one type of angular momentum, but conservation of angular momentum allows spin to convert to orbital angular momentum via spin-orbit coupling (**L** • **S**).

Science of spin currents

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The advantage of spin-orbit coupling is that atomic transitions driven by light ("electric dipole" transitions) can be used to induce a spin.

The disadvantage is that the time over which spin is a conserved quantity in solids can be rather short: (e.g., 100 ps in experiments I'll discuss). We will see consequences of this later on...

What are some other differences?

1. sensitivity to interactions 2. symmetries and Hall effect

Experimental introduction: spin drag

We outline

I. the “transient grating” method for spin dynamics

II. how it has been used to observe “spin drag” experimentally

Spin dynamics in 2D: transient grating optical measurement of spin Coulomb drag

It is difficult to access large time and length scales by neutron scattering determination of S(q,w).
An alternate method is by following the evolution of a grating of spin density:

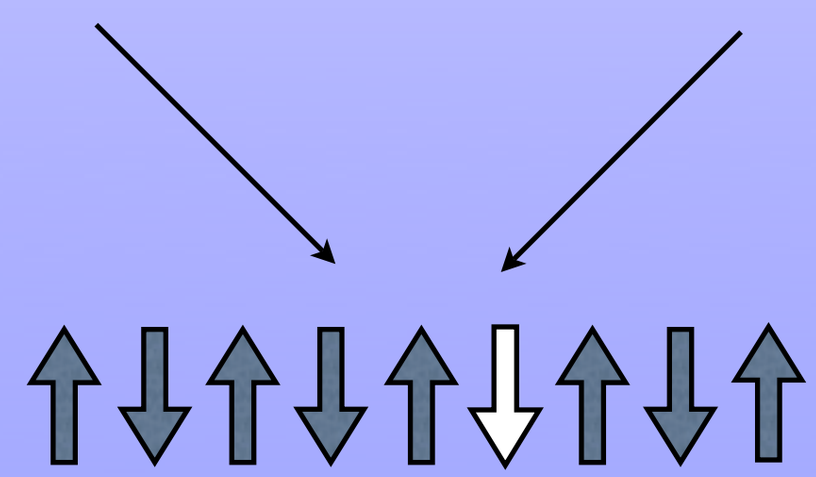
Test case:
evolution in S(q,t) of a spin density grating in high-density 2DEGs with B=0

Micron length scales and picosecond time scales: spin effectively conserved, but Coulomb interaction transfers momentum from up to down

first observation of “Coulomb drag” between up and down spins in a layer

C. Weber, N. Gedik, J. Moore, J. Orenstein, J. Stephens, D. Awschalom (Nature, 2005)

optical pump: 2 cross-polarized beams write a (circular) polarization grating



then probe in the same way (picosecond time resolution)

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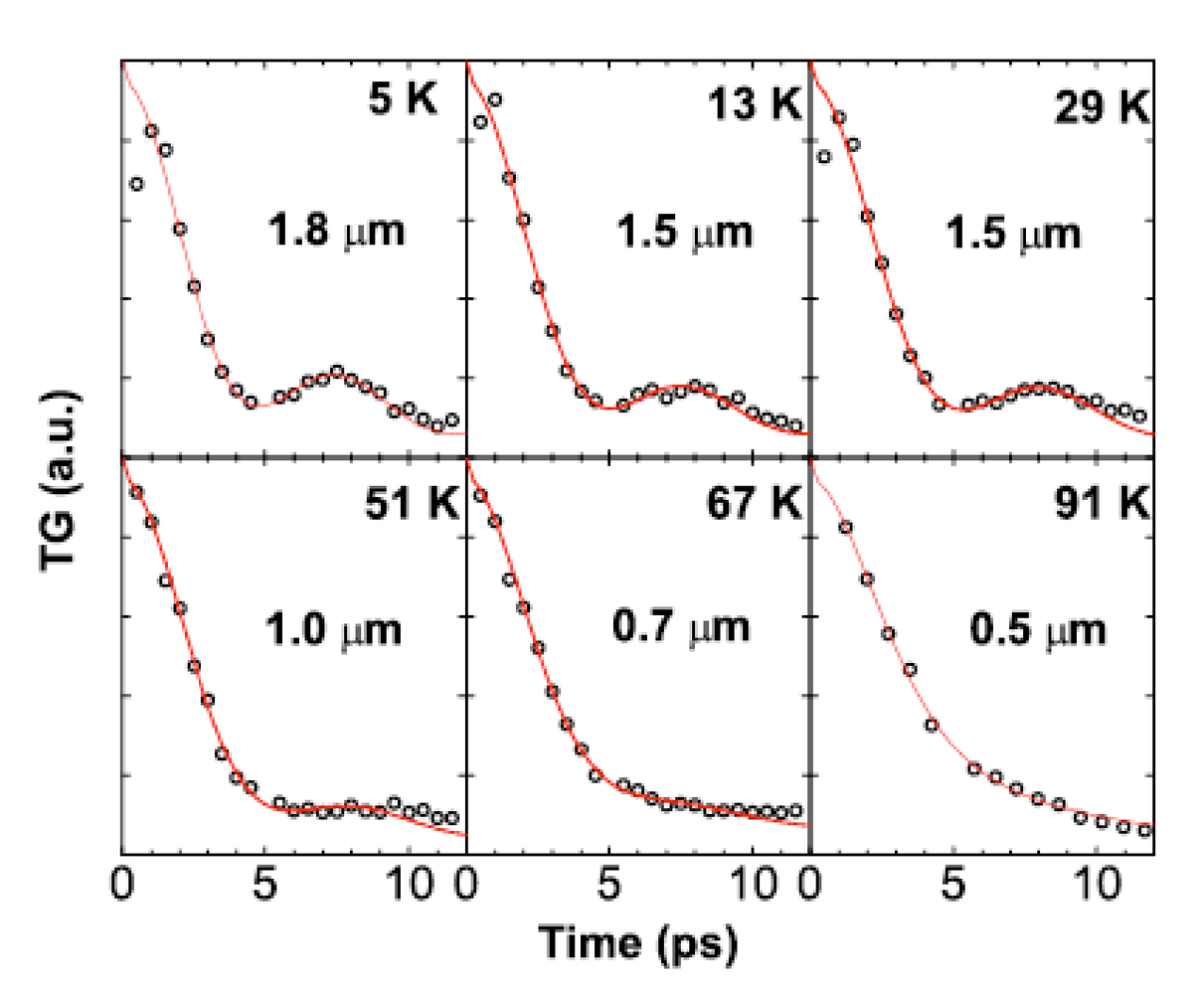

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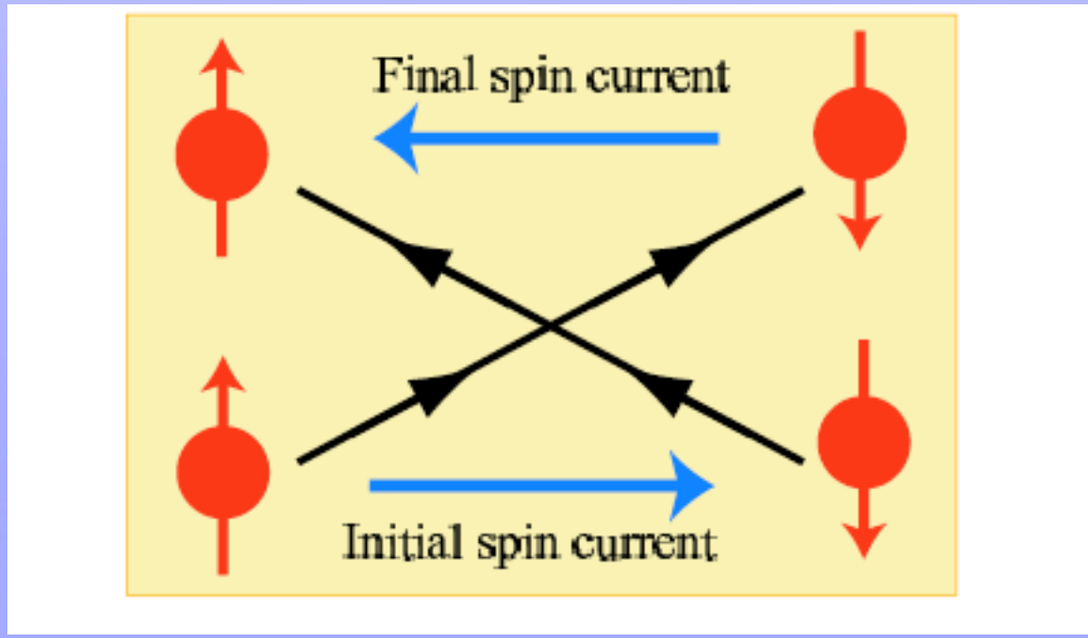
Spin dynamics in 2D: transient grating optical measurement of spin Coulomb drag

Spin drag is a fundamental difference between spin transport and charge transport, especially on short times where spin is conserved.

Momentum-conserving collisions of nearly free electrons **do not modify** the electron current, but do modify the **spin current**: the diffusion of spin density is found to be much slower than the diffusion of charge density.

(no umklapp)



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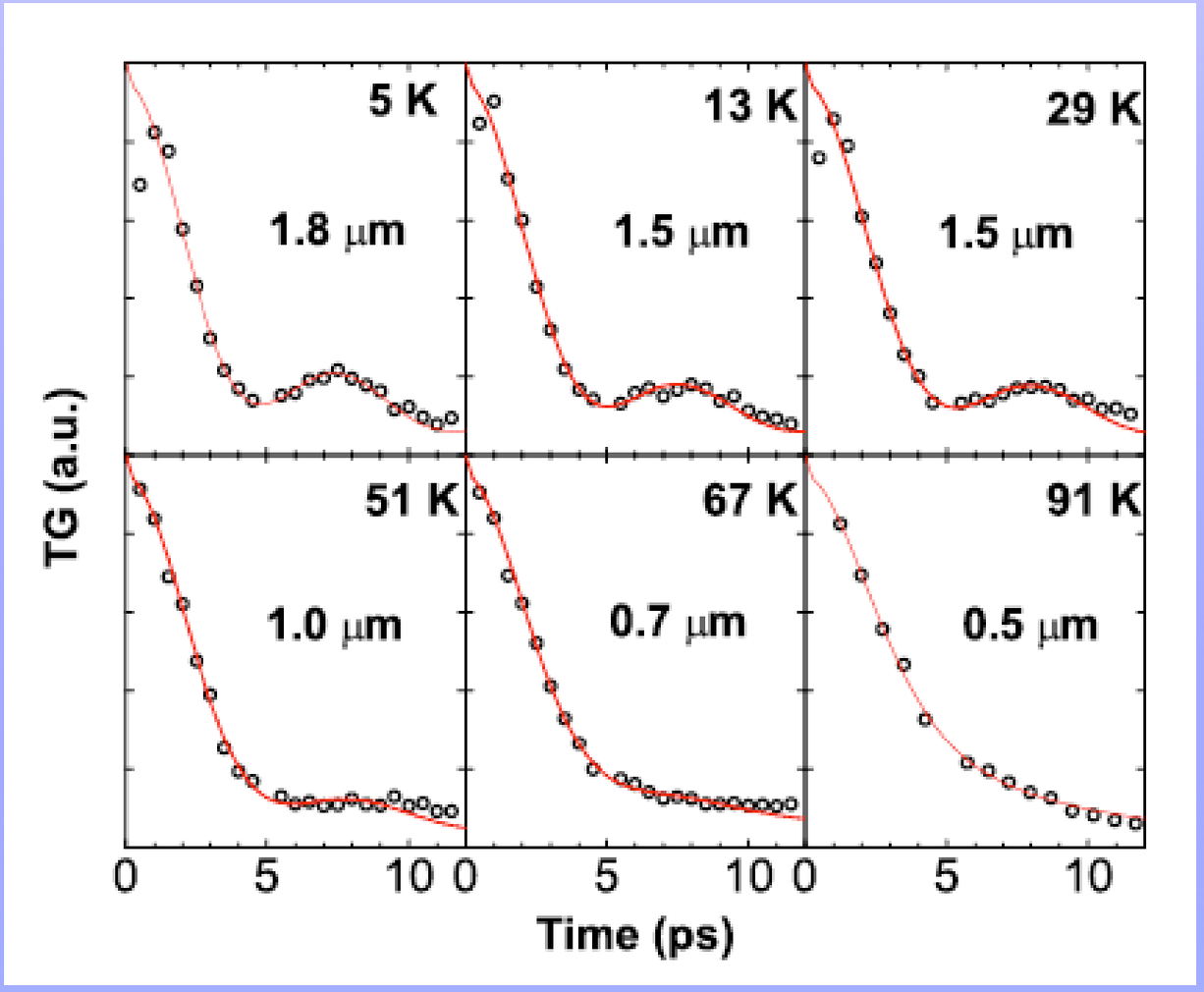
Spin dynamics in 2D: transient grating optical measurement of spin Coulomb drag

$$S(q,\omega) \propto \frac{1}{i\omega - D(q,\omega)q^2}, \quad D(q,\omega) = \frac{v_F/2}{\sqrt{(i\omega/v_F - 1/l)^2 + q^2}}$$

1. Extract spin current relaxation time by kinetic-theory fit



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1. Extract spin current relaxation time by kinetic-theory fit

$$\frac{D_s}{D_c} = \frac{1}{1 + |\rho_{\uparrow\downarrow}|/\rho}$$

2. Compare to charge current relaxation time; extract "spin drag transresistivity"

transresistivity=
(drag voltage)/(original current)

Spin dynamics in 2D: transient grating optical measurement of spin Coulomb drag

Results:

1. spin diffusion is much less rapid (factor of 5-10) than charge diffusion, at T = 300K ~ E_F

$$\frac{D_s}{D_c} = \frac{1}{1 + |\rho_{\uparrow\downarrow}|/\rho}$$

2. drag effect vanishes as T^2 at low temperatures, because phase space for scattering is reduced.

These results are consistent with theoretical predictions (D'Amico and Vignale, PRB 2001)

Even when spin is conserved, spin currents are fundamentally different from charge currents in their sensitivity to interactions

Remaining puzzle: behavior at low density

We just argued that in a metal, $D_s < D_c$.

In an insulating magnet (e.g., Hubbard model at half-filling), $D_c = 0$ but spin excitations can still propagate, so $D_s > D_c$.

Prediction: as the quantum well density is decreased, and the Coulomb repulsion drives a transition to an insulating "Wigner crystal", the sign of the "drag" effect will change.

Challenge in doing a serious calculation: the Wigner crystal transition is typically quite sensitive to disorder.

New today: using Coulomb drag between wires in a field
for spin current generation: cond-mat/0606185, Pustilnik, Mishchenko, Starykh

Spin Hall effect

Another fundamental difference between spin and charge currents is related to *symmetry*.

The ordinary Hall effect of electrons in a metal, and its quantized cousin (the *quantum Hall effect* or *QHE*), are both of the form

$$J_i = \alpha \epsilon_{ijk} E_j B_k$$

(This part I will just sketch, as it seems to have more implications for physics than chemistry at the moment)

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Under the "time reversal" operation T, velocities and currents like ***J*** change sign, as does ***B*** (but not ***E***).
(Note: this is why spontaneous magnetization is referred to as broken time-reversal symmetry.)

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Symmetry permits a dissipationless *spin Hall effect*:

$$\mathcal{J}^i_j = \sigma^s_H \epsilon_{ijk} E_k$$

because spin currents are *even* under time-reversal
(although now *inversion* symmetry must be broken).

Two classes of SHE

$$\mathcal{J}^i_j = \sigma^s_H \epsilon_{ijk} E_k$$

One mechanism for a spin Hall current is via impurity scattering (the extrinsic SHE: theory 1970s, expt. 2004)

Recent excitement has centered on the existence of an "intrinsic" SHE that arises from the band structure of a clean material.

The first proposals (Murakami et al, Sinova et al., 2003-2004) involved doped semiconductors with no gap in the band structure, but at least the most experimentally realizable of these models was found to be unstable to disorder.

Inversion symmetry breaking needed to choose a direction: in triangular n-doped GaAs quantum wells, triangular shape generates a Rashba term that drives SHE:

$$H' = \lambda(\mathbf{k} \times \sigma)_z$$

Quantized SHE

$$\mathcal{J}_j^i = \sigma_H^s \epsilon_{ijk} E_k$$

Just in the last year, different and more stable versions of the SHE have been developed that bear a close resemblance to the quantum Hall effect.

These models are believed to be more stable to disorder, on the basis of analytic arguments and explicit numerics for the case without electron interactions (D. Sheng et al., PRL 2005).

If an intrinsic SHE exists, it offers a way to create large spin currents using only an applied electric field.

The quantum Hall effect (1)

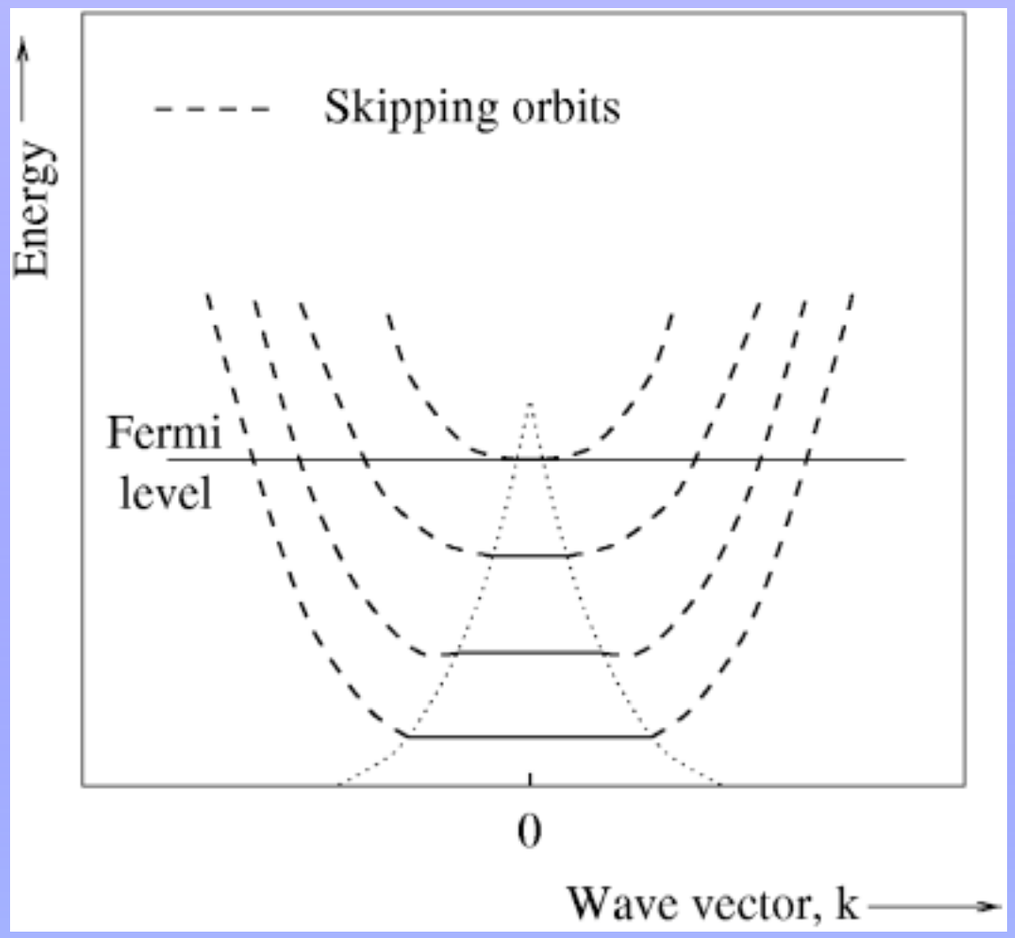
Why is the ordinary (charge) QHE so robust?

Many pictures: here is one that is useful for the SHE, based on the *edge* of the 2D Hall "droplet"

$$J_y = \frac{ne^2}{h} E_x$$

Electronic states in the bulk of the sample fall into "Landau levels" spaced by

$$\Delta E = \hbar\omega_c = \frac{\hbar e B}{mc}$$



The quantum Hall effect (3)

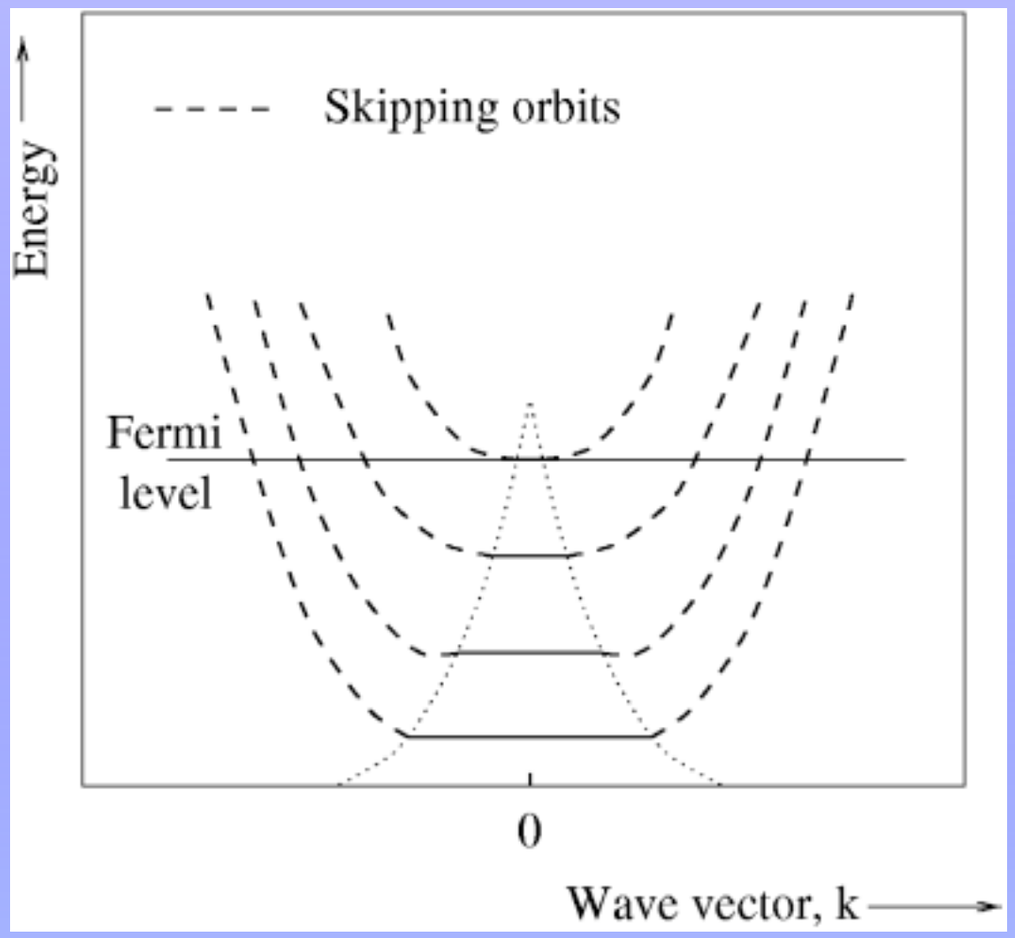
When the Fermi level is in a gap between Landau levels, the only gapless excitations that can respond to an applied field are at the edge.

Each edge is a *one-dimensional conductor*

and is *chiral*: has a specific direction (a one-way street!)

The number of levels that become gapless at the edge determines the Hall conductivity

$$J_y = \frac{ne^2}{h} E_x$$



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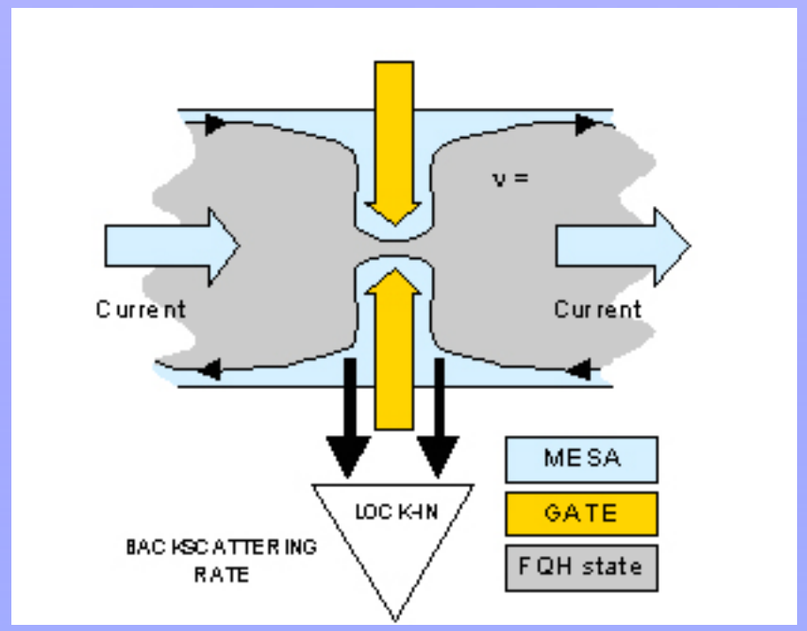

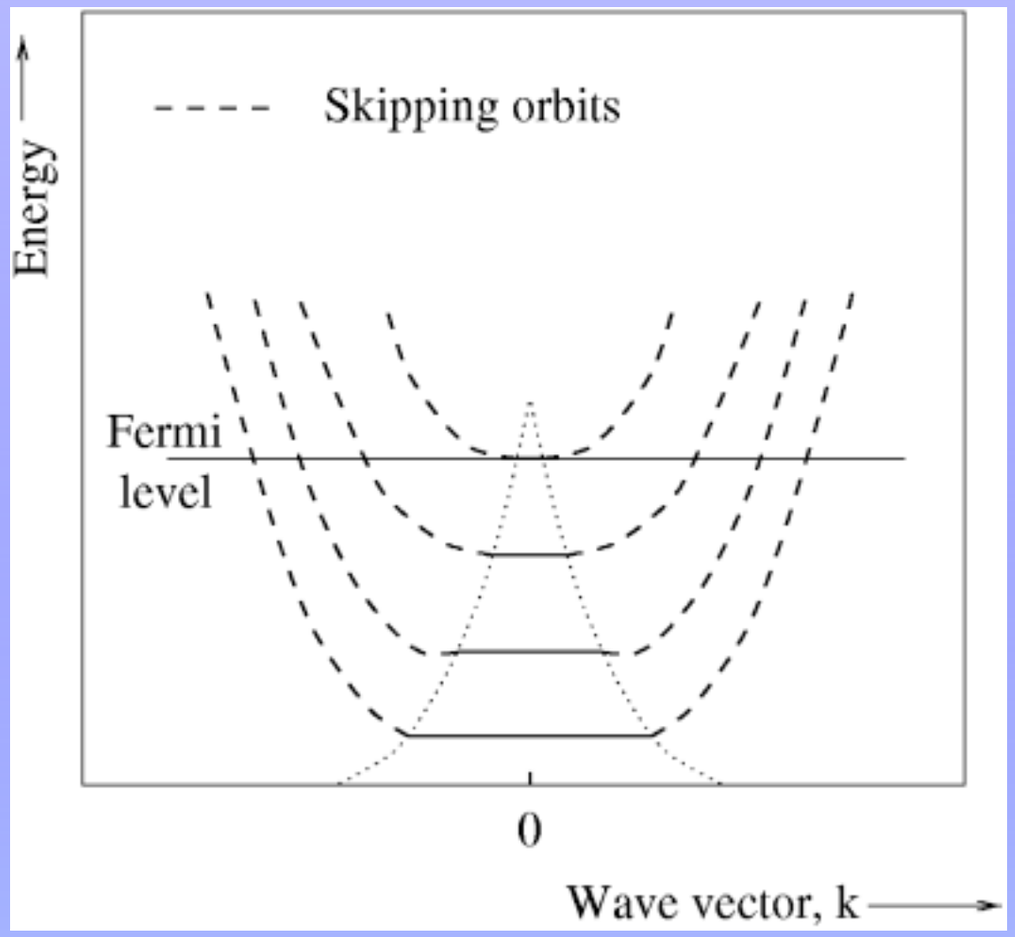

The quantum spin Hall effect

Haldane showed that although *broken time-reversal* is necessary for the QHE, it is not necessary to have a net magnetic flux.

Imagine constructing a system for which spin-up electrons feel a pseudofield along z, and spin-down electrons feel a pseudofield along -z.

Then *SU(2)* (spin rotation symmetry) is broken, but time-reversal symmetry is not:

an edge will have (in the simplest case)
a clockwise-moving spin-up mode
and a counterclockwise-moving spin-down mode

The quantum spin Hall effect

This looks very unstable: since SU(2) is broken, can't disorder scatter electrons from the spin-up edge to the spin-down edge?

It will turn out that there is an enhanced stability when there is a single pair of time-reversed edge modes (one right-mover and one left-mover): a spin-half particle cannot scatter within a time-reversed pair (a Kramers pair) if the overall Hamiltonian is *T* invariant.
(Xu and Moore, 2006; Wu, Bernevig, and Zhang, 2006)

This case seems to be realized by the QSHE in a single graphene layer (Kane and Mele, PRL 2005).
Stability to **both** disorder and interactions can be understood simply looking at the edge.

Importance beyond the SQHE

Aside from possible spin transport measurements, the deeper significance of this idea is that, in 2D, there are

exactly two topological classes of T-invariant band insulators

the "ordinary" insulator, which has an *even* number of Kramers pairs of edge modes (possibly zero)

and the "topological" insulator, which has an *odd* number of Kramers pairs of edge modes

(In 3D there are 16 classes of insulators)

Topological properties of IQHE

TKNN, 1982: the conductance is related to an integral over the magnetic Brillouin zone:

$$\sigma_{xy} = n\frac{e^2}{h}$$

$$n = \sum_{bands} \frac{i}{2\pi} \int d^2k \left(\left\langle \frac{\partial u}{\partial k_1} \middle| \frac{\partial u}{\partial k_2} \right\rangle - \left\langle \frac{\partial u}{\partial k_2} \middle| \frac{\partial u}{\partial k_1} \right\rangle \right)$$

Niu, Thouless, Wu, 1985: more generally, introducing "twist angles" around the two circles of a torus and considering the (assumed unique) ground state as a function of these angles,

$$n = \int_0^{2\pi} \int_0^{2\pi} d\theta \, d\varphi \, \frac{1}{2\pi i} \left| \left\langle \frac{\partial \phi_0}{\partial \varphi} \middle| \frac{\partial \phi_0}{\partial \theta} \right\rangle - \left\langle \frac{\partial \phi_0}{\partial \theta} \middle| \frac{\partial \phi_0}{\partial \varphi} \right\rangle \right|$$

This quantity (the "first Chern class" of the U(1) fiber bundle) is an integer. For T-invariant systems where the twists couple to the wavefunction phase, it is zero.

What about the SQHE

If a quantum number (e.g., Sz) can be used to divide bands into "up" and "down", then with T invariance, one can define a "spin Chern number" that counts the number of Kramers pairs of edge modes:

$$n_\uparrow + n_\downarrow = 0, n_\uparrow - n_\downarrow = 2n_s$$

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For general spin-orbit coupling, there is no conserved quantity that can be used to classify bands in this way. (Even if there is a way to classify bands, Sz may not be conserved, so that the spin current need not be quantized; only the number of edge modes is quantized.)

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In the bulk, one can show that the Chern number can only change by an *even* integer: each band has a Z2 invariant analogous to the integer Chern number (Moore and Balents, cond-mat).

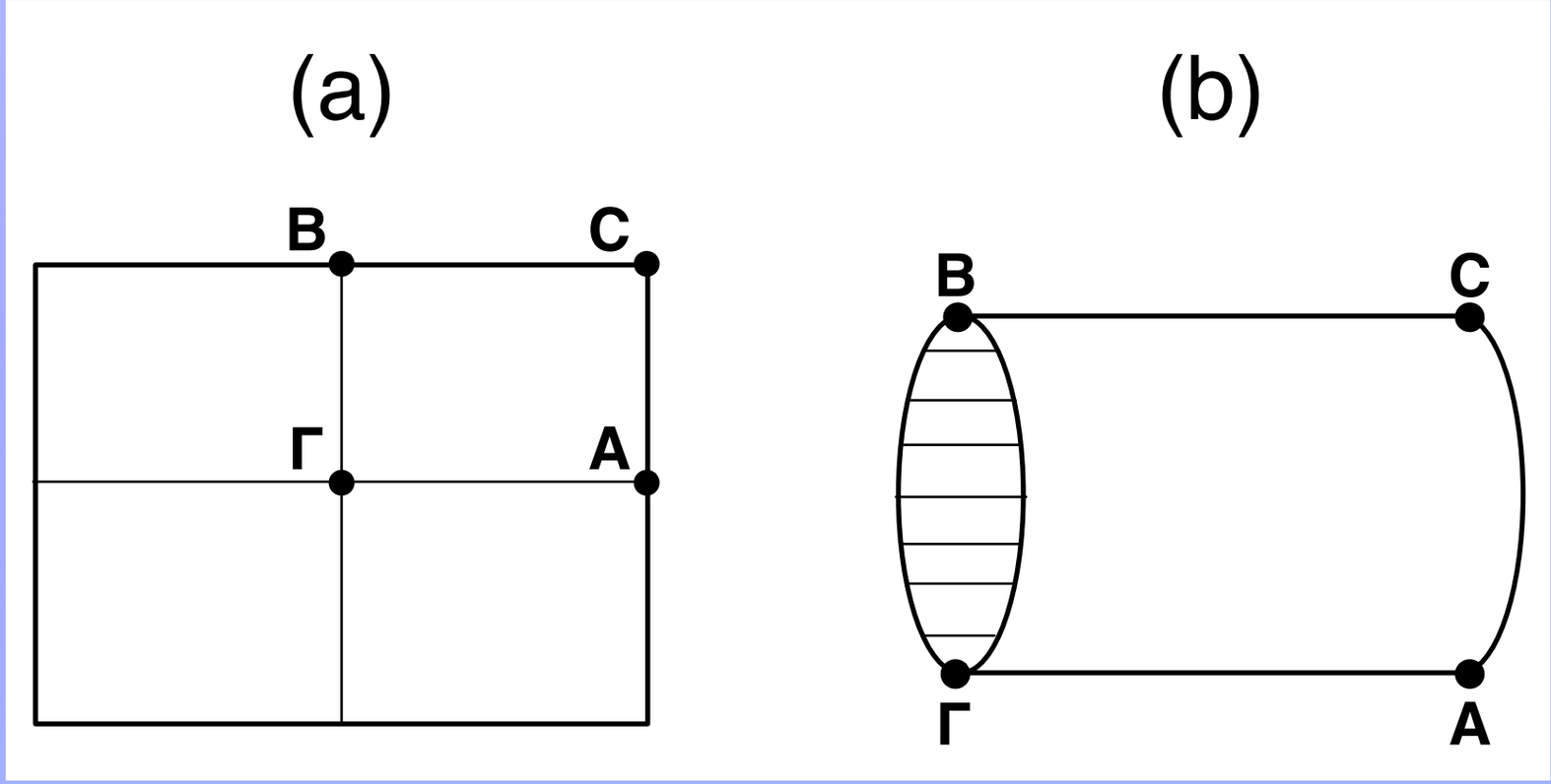
One goal of this talk is to give a physical picture of this, looking at the edge.

Z2 topological invariants

Each band of a time-reversal-invariant insulator has a Z2 invariant analogous to the integer Chern number, even when no additional quantities are conserved.
(Moore and Balents, cond-mat)

Consider a 2D Brillouin torus.

There are 4 points that are self-conjugate under time reversal (k = -k).

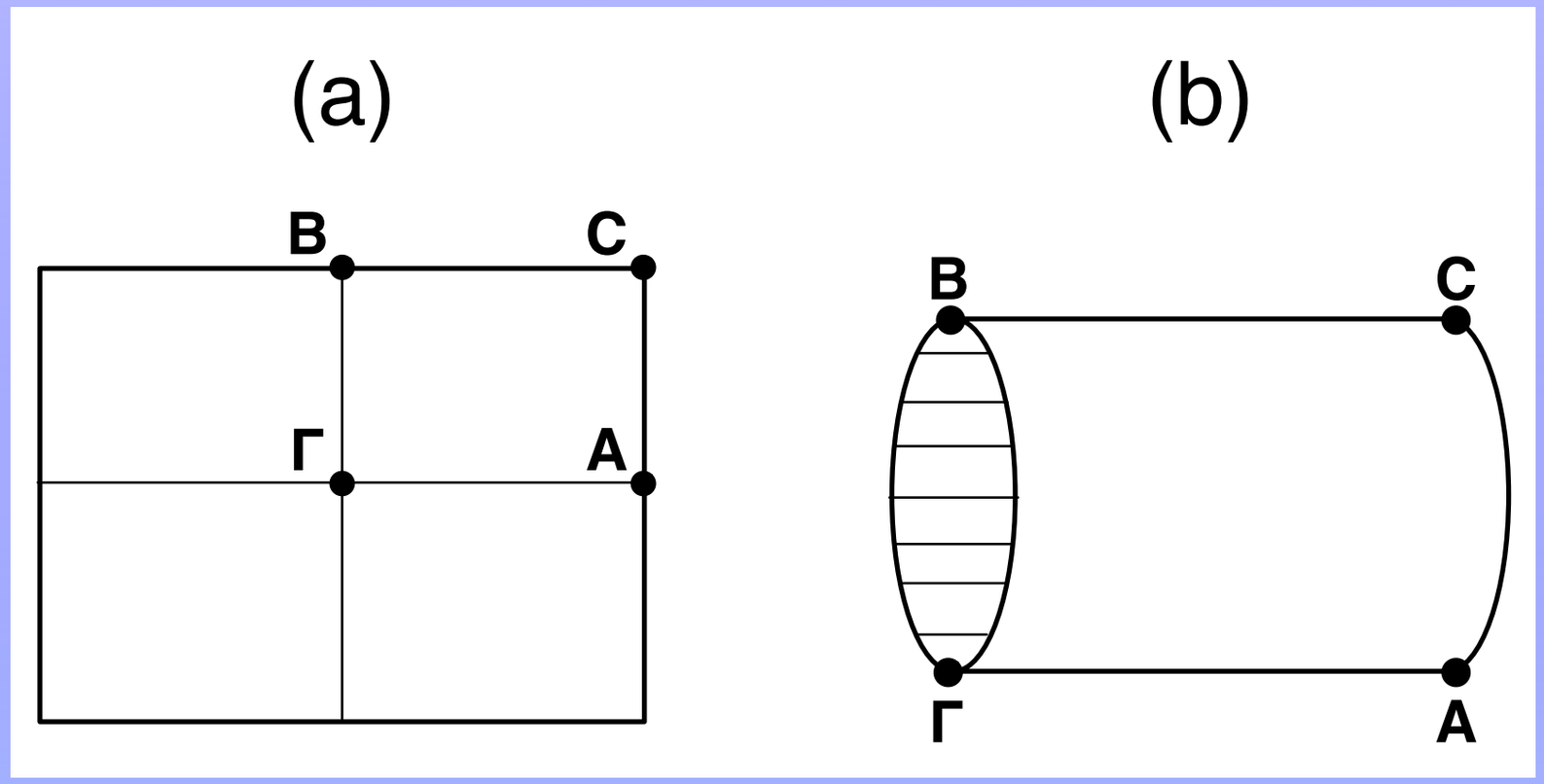


Z2 topological invariants

The Bloch Hamiltonians at points related by time-reversal are *conjugate* to each other, not necessarily identical.

$$H(-k) = TH(k)T^{-1}$$

The set of *independent* points (the effective Brillouin zone or EBZ) is then shown in (b) below: it has the topology of a cylinder with special boundary conditions.



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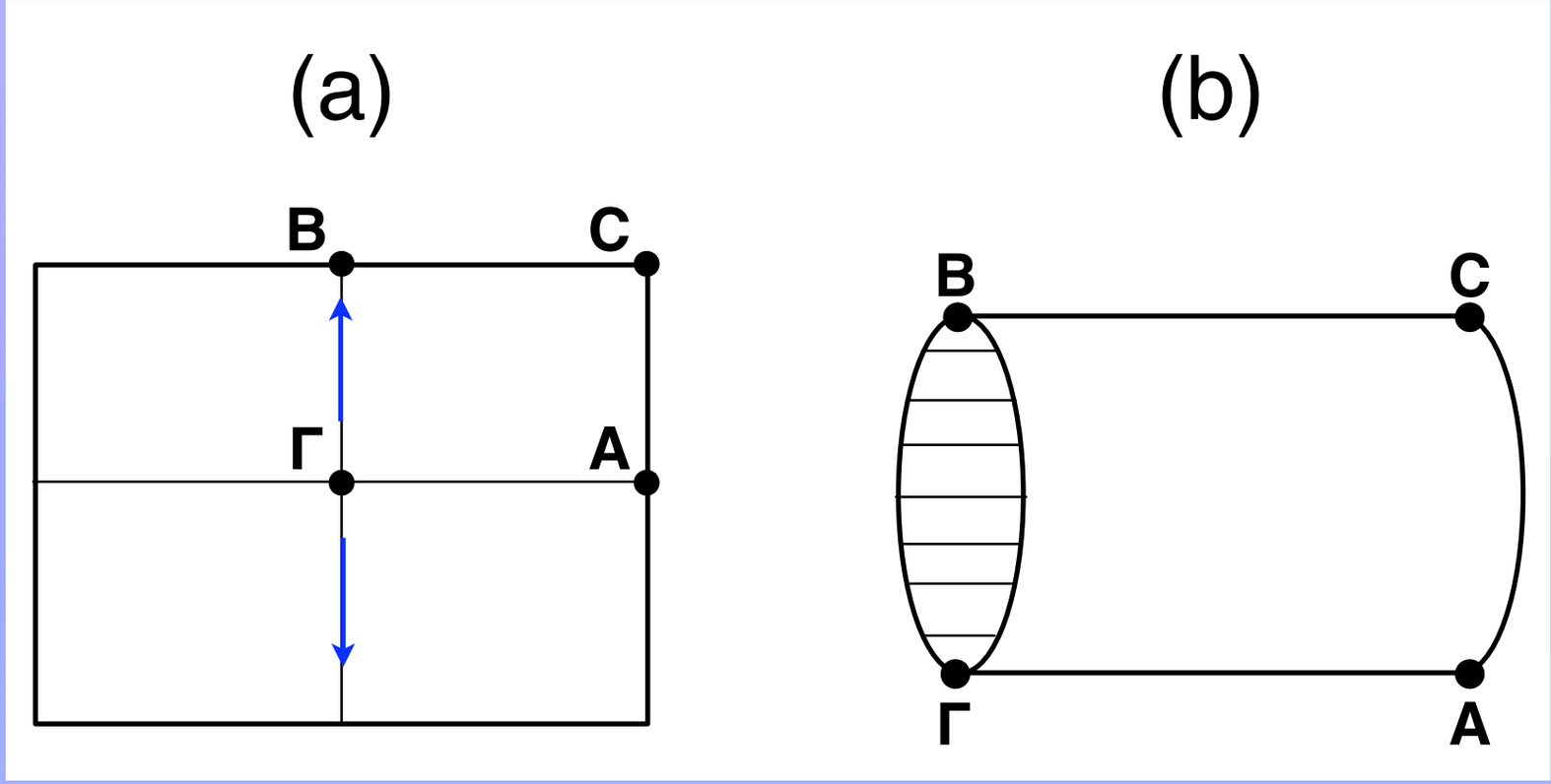


Z2 topological invariants

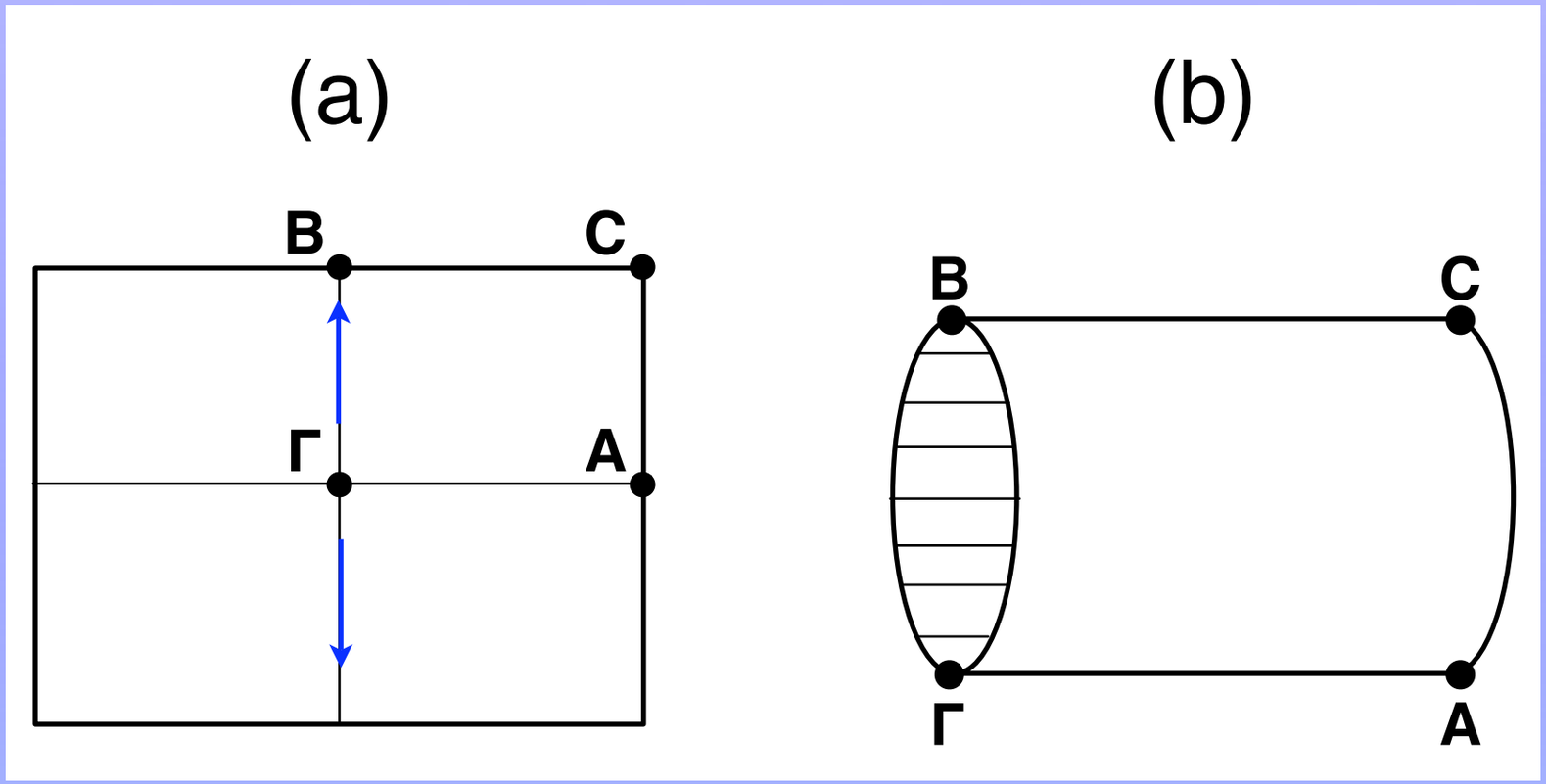
An abstract definition of the ordinary Chern number (Avron, Seiler, Simon):

if *C* is the space of Bloch Hamiltonians, then mappings from the torus to *C* are classified by *one integer for each band*, with a zero sum rule.

Why?

$$\pi_2(\mathcal{C}) = \mathbb{Z}^{n-1}, \quad \pi_1(\mathcal{C}) = 0.$$

$$\pi_n(M) = \text{equivalence classes of maps from } S^n \text{ to } M$$

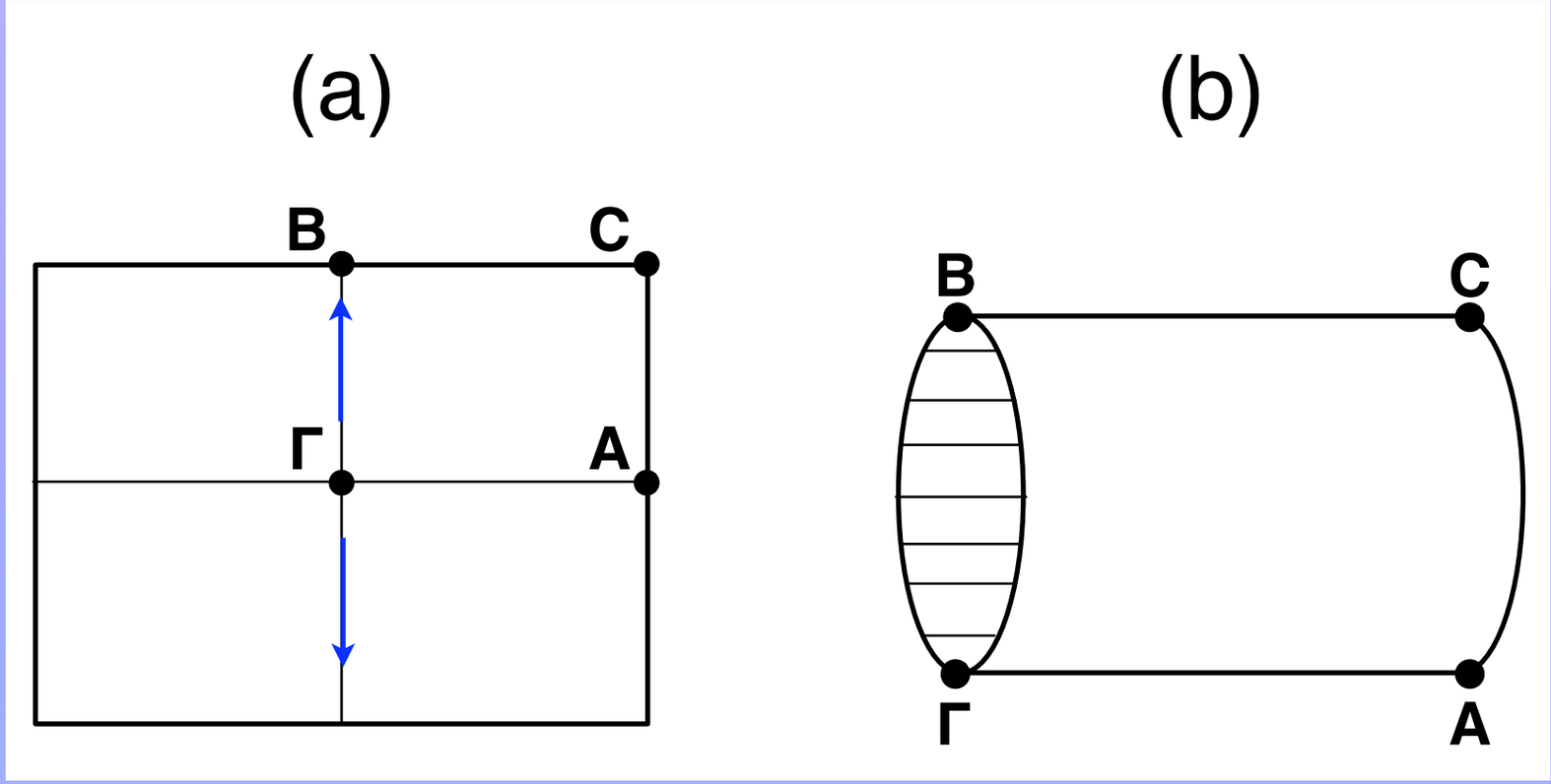


Z2 topological invariants

Key idea for the *T*-invariant case:
Consider all possible ways of "contracting" a mapping from the EBZ to one from a sphere, to define a Chern integer.

There are an *infinite* number of possible ways, differing by all *even* Chern numbers.

Need to show that the difference between two "contractions" is even:



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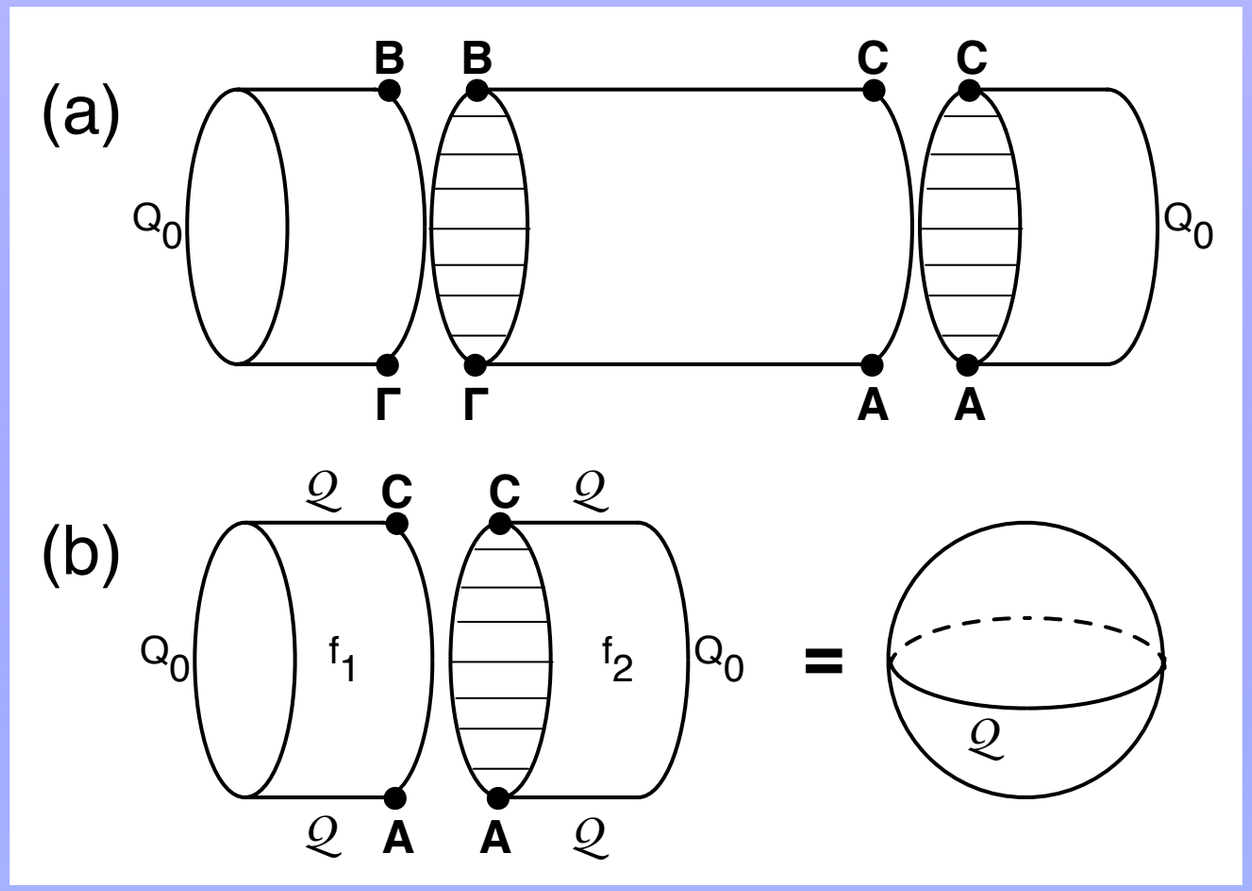
This results from the symmetries at the two circular boundaries of the cylinder.



Z2 topological invariants

Conclusions on topological classes:

there is 1 Z2 invariant (even Chern numbers or odd Chern numbers) per T-related pair of bands, with a zero sum rule;

the state of a system is the sum of Z2 invariants of occupied bands, just like the sum of ordinary Chern number determines the IQHE phase:

if the sum is odd, then the system is a *topological insulator*, with a robust spin Hall effect; if the sum is even, then the system is deformable to an ordinary insulator.

In three dimensions, there are 4 Z2 invariants per band (only a little more complicated to show).

The quantum spin Hall effect

Why are some band structures more stable than others?

Graphene = semimetal + SO coupling (creates gap)

(First observation of the QHE in graphene was in 2005!)

Edge picture: consider scattering within a T-reversed pair,

$$\langle\psi|H'|\phi\rangle = \langle T\phi|H'|T\psi\rangle = \langle\psi|H'|T^2\phi\rangle = -\langle\psi|H'|\phi\rangle$$

With interactions, multiple-particle scattering is important, and the full SHE phase diagram can be obtained using a bosonization analysis.

1. There is a wide range of stability with interactions when the Z^2 index predicts stability without interactions.

2. interactions can actually stabilize the edge, even when the Z^2 index predicts an instability in the noninteracting case (**example: 2 pairs of edge modes**).

The quantum spin Hall effect

Basic idea of bosonization (chiral and nonchiral "Luttinger liquids")

represent low-energy physics of interacting Fermi system by free Boson action

one chiral boson per propagating mode:

$$S = \frac{1}{4\pi} \int d\tau \, dx \, \partial_x \phi (\partial_\tau \phi - v \partial_x \phi)$$

Add terms that represent (spatially random) scattering of particles from one mode to another:

$$S_1 = \int dx \, d\tau \, (\xi(x) e^{i\phi_1} e^{-i\phi_2} + h.c.)$$

Key physics of spin Hall edges: the scattering process that normally leads to 1D localization is forbidden

The quantum spin Hall effect

Key physics of spin Hall edges: the scattering process that normally leads to 1D localization is forbidden

This survives for a finite strength of interactions. For sufficiently strong repulsive interactions, two-particle scattering becomes relevant and causes localization.

A surprising fact: the case with two edge modes is *unstable* in the absence of interactions, but can be stabilized by turning on a finite strength of *either* attractive or repulsive interactions
(but in the repulsive case, the tuning is delicate)

The quantum spin Hall effect

Summary: the edge of the SQHE is an unusual 1D localization problem because of the selection rule on particle scattering that results from *T*-invariance and fermionic statistics.

It is robust to weak disorder and finite interactions.
(Note that the spin current in S_z units is not quantized; it is the existence of propagating edge modes that is robust.)

Now: a few comments on when graphene is a topological insulator

Why graphene?

The spin-independent part consists of a tight-binding term on the honeycomb lattice, plus possibly a sublattice staggering

$$H_0 = -t \sum_{\langle ij \rangle} c^\dagger_{i\sigma} c_{j\sigma} + i\lambda_v \sum_i \xi_i c^\dagger_{i\sigma} c_{i\sigma}$$

The spin-dependent part contains two SO couplings

$$H' = i\lambda_{SO} \sum_{\langle\langle ij \rangle\rangle} v_{ij} c^\dagger_i s^z c_j + i\lambda_R \sum_{\langle ij \rangle} c^\dagger_i (\mathbf{s} \times \hat{\mathbf{d}}_{ij})_z c_j$$

The second (Rashba) term depends on violation of *z* mirror symmetry, and is assumed small.

The first spin-orbit term is the key: it couples to second-neighbor hopping, is mirror symmetric, and v_{ij} is ±1 depending on the sites. (It is not clear how large this term really is.)

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$$H' = i\lambda_{SO} \sum_{\langle\langle ij \rangle\rangle} v_{ij} c^\dagger_i s^z c_j + i\lambda_R \sum_{\langle ij \rangle} c^\dagger_i (\mathbf{s} \times \hat{\mathbf{d}}_{ij})_z c_j$$

Without the Rashba term, Sz is conserved, the problem decouples and a Chern number is defined for each band.

For small enough Rashba term, even though Sz is not conserved, there is still a topological classification that is much weaker: there is one Z2 (ordinary or "topological" insulator) per band, and whether the system has a spin Hall effect just depends on whether the Z2 sum is *even* or *odd*.

Quantum entanglement

Sometimes a pure quantum state of a bipartite system *AB* is also a pure state of each subsystem separately:

Example: S_z=1 state of two s=1/2 spins

$$|\Psi_{AB}\rangle = |\uparrow_A\rangle \otimes |\uparrow_B\rangle$$

a “product” state

Sometimes a pure quantum state of a bipartite system *AB* is **not** a pure state of each subsystem separately:

Example: singlet state of two s=1/2 spins

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_A\rangle \otimes |\downarrow_B\rangle - |\downarrow_A\rangle \otimes |\uparrow_B\rangle\right)$$

an “entangled” state

Quantum information

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_A\rangle \otimes |\downarrow_B\rangle - |\downarrow_A\rangle \otimes |\uparrow_B\rangle\right)$$

an “entangled” state

In an entangled state, the state of subsystem A or B is not a pure quantum state, but rather a **density matrix**

For the singlet

$$\rho_A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \rho_B$$

A classical uncertainty or **entropy** has been created by the operation of looking at only part of the system.

Quantum information

Definition: the bipartite entanglement of a pure state,
with respect to a partition into A and B,
is the von Neumann entropy of the partial density matrices

$$\langle\phi_1|\rho_A|\phi_2\rangle = \sum_j (\langle\phi_1| \times \langle\psi_j|)|\psi\rangle\langle\psi|(|\phi_2\rangle \times |\psi_j\rangle)$$

$$S(\rho) = -\mathrm{Tr}\rho_A \log_2 \rho_A = -\mathrm{Tr}\rho_B \log_2 \rho_B$$

Example: The singlet generates one bit of classical entropy when the two spins are divided

Note that the partial density matrix for subsystem A gives the results of experiments limited to A

Quantum information

Definition: the bipartite entanglement of a pure state,
with respect to a partition into A and B,
is the von Neumann entropy of the partial density matrices

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$$S(\rho) = -\mathrm{Tr}\rho_A \log_2 \rho_A = -\mathrm{Tr}\rho_B \log_2 \rho_B$$

Note that the partial density matrix for subsystem A
gives the results of experiments limited to A:

entanglement entropy is indistinguishable from any
other kind of entropy if only A is visible

CM uses of entanglement entropy

Why should you care?

Claim:

the entanglement of quantum states is related to coding efficiency, since an unentangled (product) state requires many fewer classical bits for its storage than a generic state.

Example:
how many classical real numbers are required to describe a **product** state of N spins? *O*(N)

$$|\Psi\rangle = (a_1|\uparrow_1\rangle + b_1|\downarrow_1\rangle) \otimes (a_2|\uparrow_2\rangle + b_2|\downarrow_2\rangle) \otimes \ldots \otimes (a_N|\uparrow_3\rangle + b_N|\downarrow_3\rangle)$$

vs. exponentially many for a general QM state

How much entanglement entropy occurs in ground states of real materials?

We now understand that **ground states** of "typical" local Hamiltonians generate more entropy than occurs in a product state, but far less than occurs in a generic quantum state.

Away from critical points (i.e., when correlations are short-ranged), entanglement is localized in the vicinity of the boundary.

Old result: At clean and conformally invariant quantum critical points in d=1, there is universal logarithmic entanglement with a coefficient related to the "**central charge**" of the CFT (Vidal et al.).

At criticality, the entanglement of a connected subset of N spins, with all the remaining spins, is

$$\lim_{N \to \infty} S_N = \frac{c}{3} \log N$$

Example of a critical quantum ground state:
(c=1)

$$H = J \sum_i \mathbf{s}_i \cdot \mathbf{s}_j, \quad J > 0$$

How much entanglement entropy occurs in ground states of real materials?

Some recent analytic progress (2004-present):

0. The universal log in 1D conformally invariant QCPs has a nice geometric interpretation
(Calabrese and Cardy, 2004)

1. Entanglement entropy has a log divergence with *universal* coefficient even for random spin chains in 1D: this is a universal measure of critical entropy at QCPs
(G. Refael and JEM, PRL 2004)

2. Entanglement entropy of free fermions in any dimension scales as L^{d-1} log L, i.e., violates the area law
(Gioev and Klich, PRL 2005; Wolf, PRL 2005)

3. Some 2D cases can be understood as an area law plus universal logarithmic corrections
(E. Fradkin and JEM, cond-mat 2006)

Progress toward physics goals

Entanglement entropy has been argued to underlie the success of DMRG in one dimension, and used to generate new algorithms for dynamics in 1D and statics in higher-dimensional correlated systems not amenable to other methods.
Vidal, Verstraete, Cirac, White, Schollwoeck, ...

Needed:

1. experiments to probe many-body entanglement, as a logical step en route to quantum computing

2. hard but important problems of interacting quantum spins in 2D on which to test these new algorithms!

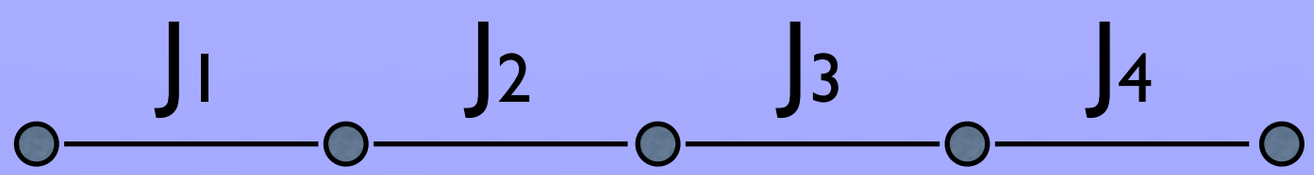
THE END.

Entanglement entropy and future directions

It turns out that even at random quantum critical points, such universal scaling exists and defines a critical entropy.

Example: **random** Heisenberg antiferromagnet (same as before, but now J on each bond is drawn from a random distribution)
(Refael and Moore, PRL 2004)

Examples include the random quantum Ising and random Heisenberg chains. The entanglement corresponds to an irrational ``central charge" but seems to satisfy some properties familiar from the clean case: a c-theorem; c(Heisenberg) = 2 c (Ising). Some of our results have been exactly confirmed numerically in work by N. Laflorencie (UBC).



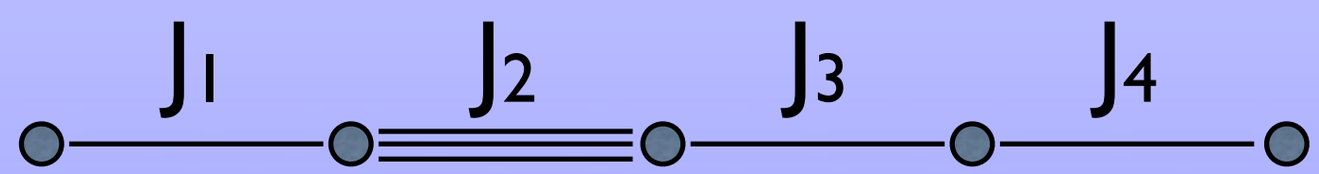
P(J) = random distribution over J > 0
(antiferromagnetic couplings)

Entanglement entropy and future directions

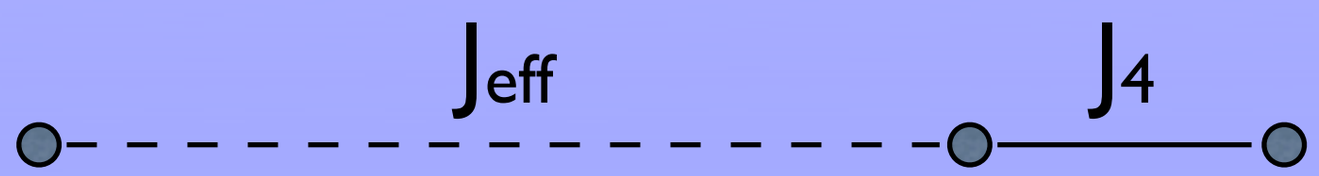
Real-space functional renormalization group for P(J):
(Dasgupta-Ma, D. Fisher)

Example: random Heisenberg chain

1. Locate strongest coupling in the chain (e.g., J_2). Form singlet between its neighboring spins to minimize its energy.



2. Remove singlet from low-energy theory; compute effective residual coupling between next-nearest-neighbor spins.



$$J_{\text{eff}} \sim \frac{J_1 J_3}{2 J_2}$$

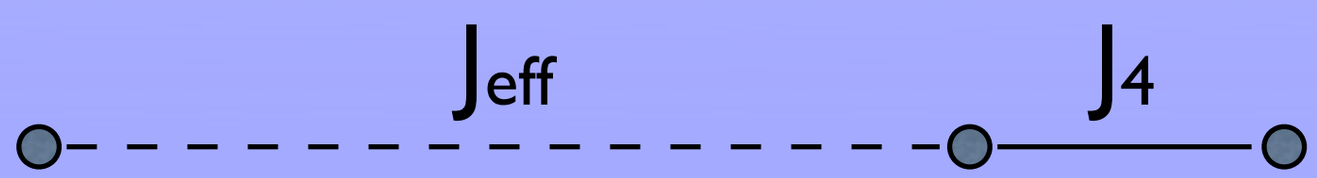
Entanglement entropy and future directions

This process of forming singlets and effective couplings generates the following functional RG equation for the coupling distribution:

$$\frac{\partial \rho(\xi,\Gamma)}{\partial \Gamma} = \frac{\partial \rho}{\partial \xi} + \rho(0,\Gamma)\int_0^\infty d\xi_- \int_0^\infty d\xi_+ \, \delta(\xi - \xi_+ - \xi_-)\rho(\xi_+,\Gamma)\rho(\xi_-,\Gamma)$$

The theory is self-consistent if the final distribution on large length scales is very **broad**, so that the largest coupling is typically much larger than its neighbors.

Then the second-order perturbation theory result used to generate the RG equation is justified.



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Entanglement entropy and future directions

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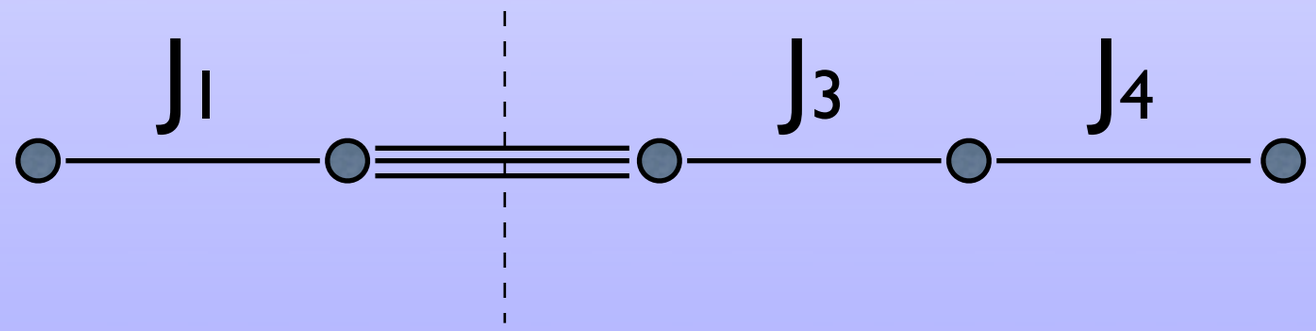

For entanglement entropy, we need to obtain the mean number of singlets that form across a boundary.

The main subtlety in the calculation is a "memory" effect: after decimation, the newly created effective J is **weaker** than an average J, and hence less likely to be decimated.

This corresponds to a repulsion between decimation events in RG time, which gives a factor of 1/3 correction to a simple estimate.

Entanglement entropy and future directions

Numerical confirmation of **irrational** central charge for one case (XX chain->free fermions) by N. Laflorencie, UBC (PRB 2005)

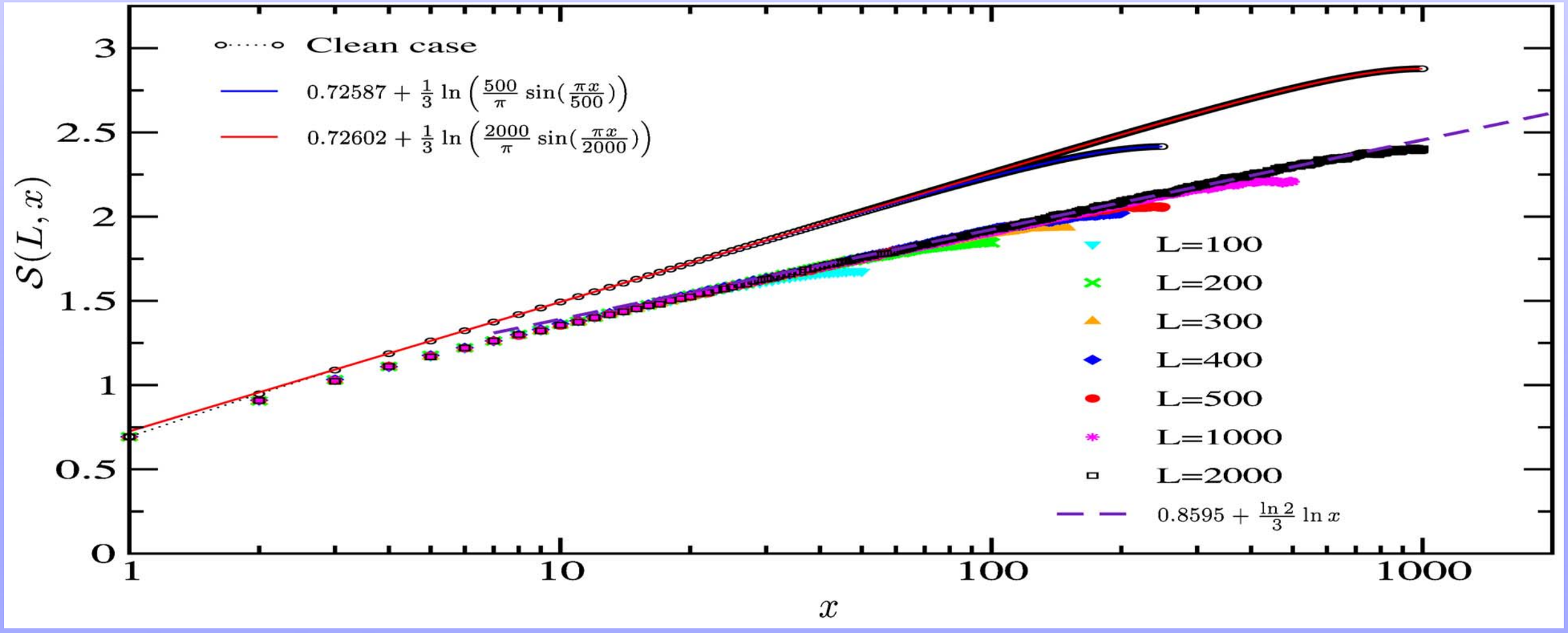

FIG. 2. (Color online) Entanglement entropy of a subsystem of size x embedded in a closed ring of size L, shown vs x in a log-linear plot. Numerical results obtained by exact diagonalizations performed at the XX point. For clean nonrandom systems with $L=500$ and $L=2000$ (open circles), $\mathcal{S}(x)$ is perfectly described by Eq. (3) (red and blue curves). The data for random systems have been averaged over 10^4 samples for $L=500$, 1000, 2000, and 2×10^4 samples for $100\leqslant L\leqslant 400$. The expression $0.8595+(\ln 2/3)\ln x$ (dashed line) fits the data in the regime where finite size effects are absent.

Entanglement entropy and future directions

The upshot is that we now believe that essentially any **local** Hamiltonian generates only short-ranged entanglement in the ground state.

This motivates a new class of numerical methods: search for correlated ground states variationally within a class of locally entangled states (e.g., "matrix product states", Ostlund and Rommer, 1995).

It has even been proven in some cases that the ground state is well described by some matrix product state, **but**

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It has even been proven in some cases that the ground state is well described by some matrix product state, **but**

the remaining computational challenge is to ensure that the search process does converge to the optimal matrix product state.

This could advance greatly theoretical understanding of some of the models mentioned earlier: for example,

frustrated magnets and magnetic molecules
fermionic correlated systems (cuprates and other Mott insulators)

Entanglement entropy and future directions

Why is **c**, the "central charge", so important physically in conformal field theories? (mathematical definition is via Virasoro algebra or OPE of stress-energy tensor)

It is a standard measure of entropy at critical points: for example, a quantum critical point in 1D described by a CFT of central charge c has low-temperature free energy per length (Affleck 1986)

$$f = \frac{F}{L} = f_0 - \frac{\pi}{6} c (kT)^2 \hbar v$$

A fundamental property of **c** is the "c-theorem":
if a relevant operator at critical point A takes the system to critical point B, then

$$c_B \leq c_A$$

The decrease of **c** along RG trajectories is consistent with the interpretation of RG as "integrating out" degrees of freedom.

Applications to condensed matter theory

What about entanglement in higher dimensions?

Free fermions can show either a pure area law (L^{d-1}) or an additional logarithm (L^{d-1} log L) depending on the nature of the Fermi surface (Wolf; Klich and Gioev). (Fermions are actually quite special!) Note that in both these cases, there must be a dimensionful (nonuniversal) scale factor.

For general critical points, the answer is unknown.

With E. Fradkin and M. Negrete, we have recently obtained the entanglement entropy for "conformal quantum critical" points in 2D: (e.g., quantum dimer model).

We find universal logarithmic corrections for some geometries.

Historical note: the original appearance of the entanglement entropy area law was to understand why black holes have "holographic" entropy (S ~ A, the area of the event horizon).
(Bombelli et al.; 1986 Srednicki 1993)

2D conformal quantum critical points

Example: critical point of quantum dimer model
Hilbert space basis: classical dimer coverings of square lattice

$$H = -t(\text{flip plaquettes with parallel dimers}) + V(\text{count flippable plaquettes})$$

$$H = -t \sum \left(|\!\!=\!\!\rangle\langle\!|\ \ |\!| + |\!|\ \ |\rangle\langle\!\!=\!\!| \right) + V \sum \left(|\!\!=\!\!\rangle\langle\!\!=\!\!| + |\!|\ \ |\rangle\langle\!|\ \ |\!| \right)$$

$$H = \begin{pmatrix} n_1 V & -t & 0 & 0 & -t & \dots \\ -t & n_2 V & -t & 0 & 0 & \dots \\ 0 & -t & n_3 V & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

2D conformal quantum critical points

Resulting form of Hamiltonian matrix:

$$H = \begin{pmatrix} n_1 V & -t & 0 & 0 & -t & \dots \\ -t & n_2 V & -t & 0 & 0 & \dots \\ 0 & -t & n_3 V & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Here the diagonal terms counts the number of flippable plaquettes, which is also the number of nonzero off-diagonal elements

At t=V, equal-weight superposition is an exact E=0 eigenstate: quantum critical wavefunction with correlations given by classical critical model (dimer packings).

Continuum wavefunction: one free boson (c=1 CFT)

$$|\psi\rangle = e^{-S_E(\{\phi\})/2} |\{\phi\}\rangle$$

2D conformal quantum critical points

Result I: von Neumann entropy of such a wavefunction under a partition into A and B is determined by free energy in the CFT:

$$S = F_A + F_B - F_{A\cup B}$$

Here the first two terms are with Dirichlet boundary conditions at the AB boundary.

Now use M. Kac result on "hearing the shape of a drum": for a 2D connected region with smooth boundary,

$$F_A \sim f_0(L/a)^2 + f_s(L/a) - \frac{c\chi}{6}\log(L/a)$$

Euler characteristic

$$\chi = 2 - 2g - b$$

2D conformal quantum critical points

Result II: the area law contribution is exactly determined by the boundary energy in the CFT with the same regularization

Result III: there is a universal logarithmic corrrection in the following two cases:

I. the boundary "cuts" the system into pieces and thus modifies the total Euler characteristic (example: cut disk into 2 pieces)

II. the boundary has sharp corners (e.g., A is a square within B)

$$S = F_A + F_B - F_{A\cup B}$$

$$F_A \sim f_0(L/a)^2 + f_s(L/a) - \frac{c\chi}{6}\log(L/a)$$

Euler characteristic

$$\chi = 2 - 2g - b$$

Collaborators:
Experiment: C. Weber, N. Gedik, J. Orenstein (UCB)
J. Stephens, D. Awschalom (UCSB)

Theory: G. Refael (Caltech)
M. Negrete and E. Fradkin (UIUC)
N. Bray-Ali, S. Mukerjee, D. Podolsky, C. Xu,
D.-H. Lee, and A. Vishwanath (UCB)

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THE END

The quantum spin Hall effect

Why are some band structures more stable than others?

Graphene = semimetal + SO coupling (creates gap)

(First observation of the QHE in graphene was in 2005!)

Edge picture: consider scattering within a T-reversed pair,

$$\langle\psi|H'|\phi\rangle = \langle T\phi|H'|T\psi\rangle = \langle\psi|H'|T^2\phi\rangle = -\langle\psi|H'|\phi\rangle$$

where we have used the antiunitarity of T and the relations (K is conjugation)

$$T^\dagger H' T = H', \quad T = U_\tau K, \quad T^\dagger = T^{-1} = K U_\tau^{-1}$$

The first equality follows from

$$\langle\psi|H'|\phi\rangle = \langle H'\psi|\phi\rangle = \langle T\phi|TH'\psi\rangle = \langle T\phi|H'|T\psi\rangle$$

then $T\phi = \psi$ gives the second, and T^2=-1 for spin-half particles in the last.

Entanglement entropy and future directions

The irrational values of "effective **c**" found via entanglement entropy at random critical points are consistent with c-theorem ideas:

disorder is **relevant** at the clean critical point, and the RG flow is to the random critical point, with reduced $c_{eff} = (\ln 2)\, c_0$.

Are there other cases where an effective c-theorem holds for quantum critical points?

We are currently studying a generalization of the problem: higher-spin integrable models may flow under randomness to permutation-symmetric random critical points; is the c-theorem satisfied there as well?

$$c = \frac{3s}{s+1}$$

$$\tilde{c} = ???$$

Spin

s=1/2 $\quad c = 1$

$$\tilde{c} = \ln 2$$

Disorder

Quantum information and black holes

Historical note: the original appearance of the entanglement entropy area law was to give a picture why black holes have "holographic" entropy (S ~ A, the area of the event horizon).
(Bombelli et al.; 1986 Srednicki 1993)

Argument:
let subsystem A be the black hole; subsystem B be the rest of the universe.

Assume that the whole universe is in a pure state. What entanglement entropy is found by an observer that cannot access the black hole?

Guess I: entanglement entropy is extensive in "spatial volume" (assuming that GR provides a unique definition of the latter).

But subsystem A is finite, subsystem B is infinite (or at least much larger).

Guess II: entanglement entropy should be determined by a property shared by A and B: the boundary (event horizon) -> holographic entropy.

The quantum spin Hall effect

The extra stability with an odd number of Kramers pairs at the edge is fundamentally a "Z^2 topological index" of the band structure in the bulk.

However, with interactions two right-moving particles can interact with each other and scatter jointly into two-left-moving particles: does this modify the QSHE?

A bosonization analysis gives the following conclusions:

1. There is a wide range of stability with interactions when the Z^2 index predicts stability without interactions.
2. interactions can actually stabilize the edge, even when the Z^2 index predicts an instability in the noninteracting case.